A mathematical study on squeeze film lubrication between parallel plates

Dissertation submitted to
Visvesvaraya National Institute of Technology, Nagpur
in partial fulfillment of the requirement for the award of the degree

Master of Science

in

Mathematics

by

Latika
(MS23MTH008)

under the guidance of

Dr. M. Devakar

Department of Mathematics
Visvesvaraya National Institute of Technology
Nagpur - 440 010 (India)
May 2025

A mathematical study on squeeze film lubrication between parallel plates

Dissertation submitted to
Visvesvaraya National Institute of Technology, Nagpur
in partial fulfillment of the requirement for the award of the degree

Master of Science

in

Mathematics

by

Latika
(MS23MTH008)

under the guidance of

Dr. M. Devakar

Department of Mathematics
Visvesvaraya National Institute of Technology
Nagpur - 440 010 (India)
May 2025

©Visvesvaraya National Institute of Technology (VNIT) 2025

Department of Mathematics
Visvesvaraya National Institute of Technology Nagpur

Declaration

I, Latika, hereby declare that this dissertation titled "A mathematical study on squeeze film lubrication between parallel plates" is carried out by me in the Department of Mathematics of Visvesvaraya National Institute of Technology, Nagpur. The work is original and has not been submitted earlier whole or in part for the award of any degree/diploma at this or any other institution/University.

Name and Signature

Date:

Certificate

This is to certify that the dissertation entitled "A mathematical study on squeeze film lubrication between parallel plates" is a bonafide work done by **Latika** (Enrollment No. MS23MTH008). The project work has been completed under my guidance and supervision in a satisfactory manner. This work is a part of the requirement for the award of **Master of Science in Mathematics** from Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur. This dissertation fulfills the requirements relating the nature and standard of work for the award of Master of Science degree.

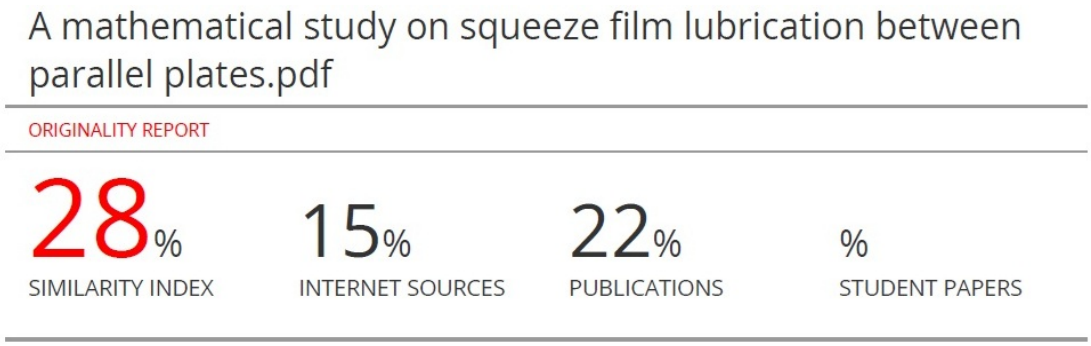
Dr. M. Devakar
Professor
Dept. of Mathematics, VNIT Nagpur

Head, Department of Mathematics
VNIT, Nagpur
Date:

VISVESVARAYA NATIONAL INSTITUTE OF TECHNOLOGY, NAGPUR

Non - Plagiarism Certificate

Certified that the M.Sc. dissertation titled "A mathematical study on squeeze film lubrication between parallel plates" submitted by **Latika**, Enrollment No. MS23MTH008, ID No. 30139 has been checked for plagiarism using TURNITIN software and the overlap is found within prescribed limits. A summary of the report is given below:



Signature:

Student Name: Latika

Enrollment No: MS23MTH008

ID No: 30139

Signature of Supervisor

Acknowledgement

It is of great pleasure and proud privilege to express my deep sense of gratitude to my supervisor Dr. M. Devakar, Professor, Department of Mathematics, Visvesvaraya National Institute of Technology (VNIT), Nagpur for his guidance and generous encouragement throughout my project work. Without his guidance, it would have not been possible to complete this work.

I convey my deep sense of gratitude to Dr. Pallavi Mahale, Professor and Head, Department of Mathematics, VNIT Nagpur for her invaluable co-operation and excellent facilities throughout the session. I thank project progress and evaluation committee members Dr. Pradip Roul and Dr. Vishnu Pratap Singh, Assistant Professors of Department of Mathematics for their encouragement during progress of the work. I am also thankful to all faculty members of the Department of Mathematics, VNIT Nagpur for their continuous support.

I am grateful to Mr. Prabal Acharya and Mr. Nikhil Kumar, Ph.D Research Scholars of the Department, for their valuable time, suggestions, and discussions provided at each step of the work.

I express my sincere thanks with gratitude to my parents and other family members for their extended support, blessings and encouragement.

Finally, I bow down before the almighty that has made everything possible.

Latika

Abstract

This dissertation presents a theoretical study of fluid dynamics relevant to squeeze film lubrication. The foundational principles of fluid motion namely, the conservation of mass, linear momentum, and energy are presented. From these fundamental laws, the continuous equation, Navier–Stokes equations and energy equation are systematically derived. The classical problem of squeeze film lubrication between two parallel plates is then formulated by incorporating inertia effects, extending beyond the traditional lubrication approximation. An analytical solution is obtained under appropriate assumptions, providing explicit expressions for the pressure distribution within the fluid film. The influence of inertia is analyzed by varying the Reynolds number, revealing its effect on load-carrying capacity and pressure profiles. It is found that the quantitative effects of fluid inertia on the improved squeeze film characteristics become more pronounced with an increase in the Reynolds number and a decrease in the squeeze film height.

Contents

Abstract **i**

1 Introduction **1**

2 Conservation laws and their mathematical form **5**

2.1 Law of conservation of mass 5

2.1.1 Mathematical form of law of conservation of mass – Continuity equation 6

2.2 Law of conservation of linear momentum 7

2.2.1 Mathematical form of law of conservation of linear momentum – Navier-Stokes equations 7

3 Squeeze film lubrication between parallel plates **11**

3.1 Mathematical formulation 12

3.2 Solution 13

3.3 Discussion of results 19

3.4 Conclusions 26

4 Conclusions **27**

Chapter 1

Introduction

Lubrication

It is known since ages that liquids and greases reduce friction between sliding surfaces by filling the surface cavities and making the surfaces smoother. The action of liquid or grease is known as **lubrication** [1]. In other words, lubrication is a process by which the friction and wear rate in a moving contact are reduced by using a suitable lubricant. Almost every assembly consisting of relatively moving components requires lubrication. Lubrication is essential for ensuring smooth operation and longevity of mechanical components in various applications such as bearings, gears, locks, and other moving parts.

Lubricant is a substance introduced between relatively moving parts to reduce friction and wear rate.

Mechanism of fluid flow in Lubrication

A fluid can reduce friction and wear between two relatively moving surfaces when a thin film of lubricant is applied. Normally, the actual contact area between relatively moving surfaces is very small, leading to high stress concentrations. Without lubrication, these stress levels can cause plastic deformation in ductile materials and cracking in brittle materials, leading to wear.

The lubricant helps by redistributing the load, reducing normal stresses, and providing a low shear force, which helps smoother relative motion.

Types of fluid film lubrication

Fluid film lubrication can be classified as:

1. **Hydrodynamic lubrication**
    - In this mechanism, a fluid is drawn between moving surfaces due to relative velocity.
    - If the lubricant is a liquid, the mechanism is called hydrodynamic lubrication.
    - Viscosity of lubricant plays an important role to support the load and reduce friction.

2. **Squeeze film lubrication**
    - It occurs when there are fluctuations in load and/or relative speed between two surfaces.
    - It generates a squeeze film action, where the lubricant is compressed between the surfaces.
    - Viscosity plays an important role in this type of lubrication.

3. **Hydrostatic lubrication**
    - In this type, an external pressure is applied to supply fluid between the surfaces.
    - This creates a lubrication film without requiring relative motion between the surfaces.

More about squeeze film lubrication

Squeeze film behaviour is widely observed in many applications such as machine tools, gears, bearings, rolling elements, automotive engines, hydraulic systems and skeletal joints. The squeeze film phenomena arise from the behaviour of two lubricated surfaces approaching or receding each other with a normal velocity. When the surface approach each other, positive squeeze arises, while receding surfaces can generate negative squeeze. For a positive squeeze, the fluid is squeezed out like a jet, whereas, for negative squeeze, the surrounding fluid is sucked into the gap between the plates along the streamlines and the fluid is contracted rapidly while entering the gap between the plates. In case of positive squeeze $\frac{dh}{dt} < 0$, and $\frac{dh}{dt} > 0$ for negative squeeze, where $\frac{dh}{dt}$ is the relative velocity of the two surfaces and h is film thickness. Since the viscous lubricant present in the film has resistance, it cannot be squeezed out instantaneously.

Brief literature

Wu [2] have analyzed the squeeze film behaviour between porous rectangular plate. Ramanaiah [3] has studied the effect of couple stress fluid in squeeze film lubrication between finite plates. Naduvinamani et al. [4] and Lin et al. [5] have investigated the effect of couple stress fluid in squeeze film lubrication between different geometry. In all these studies, some advantages over the Newtonian lubricants have been predicted such as increased load carrying capacity, and delayed time of approach. Lin [6] have analyzed the effect of magnetohydrodynamics in squeeze film characteristics for finite rectangular plates. Bujurke and Kudenatti [7], and Kudenatti et al. [8] have studied the effect of MHD in squeeze film lubrication between different geometries. It is observed that the effect of magnetic field is to provide a significant load carrying capacity and ensure a delayed squeezing time compared to the classical case. Hanumagowda et al. [9] and Patil et al. [10] have studied the combined effect of MHD and couple stress fluid.

In all these studies, authors have considered that the effect of lubricant inertia is small compared with the lubricant viscosity. In recent development, lubricants with low or high viscosity(such as synthetic lubricants) and with high velocities are also used in industries. Because of high velocity, it is possible to attain such a situation where flow is laminar but fluid inertia forces are still significant. Hence, the classical assumption of ignoring inertial effects may not properly model such situation with increasing Reynolds number. Therefore, fluid inertia effect should be considered in these situation.

In this dissertation, the squeeze film motion between two parallel plates has been analytically investigated with the inclusion of inertia effects, which are often neglected in classical lubrication theory. The following lubrication assumptions have been used for said study:

1. The flow in the film is laminar.
2. The fluid is incompressible and has constant properties.
3. Pressure is independent along the thickness.
4. $\varepsilon = \frac{h_0}{l_c} << 1$, where h_0 is thickness between plates and l_c is characteristic length.

This dissertation is organized in four chapters. The first chapter provides a brief overview of lubrication theory along with literature survey. The second chapter presents the derivation of governing equations, including the continuity, momentum and energy equations. The third chapter focuses on the core work of the current dissertation — the analysis of squeeze film motion between two parallel plates with consideration of inertia effects. The expression for pressure distribution and load carrying capacity are derived analytically and their dependence on Reynolds number is discussed. Finally, the fourth chapter summarizes the findings of the study along with directions for future work.

Chapter 2

Conservation laws and their mathematical form

The focus of this chapter is to derive mathematical equations concerning the governing laws of fluid dynamics, namely, the law of conservation of mass and the law of conservation of linear momentum.

2.1 Law of conservation of mass

Law of conservation of mass states that mass is neither being created nor destroyed, it remains constant over time. More precisely, when a region of fluid contains neither sources (inlets) nor sinks (outlets) through which fluid can enter or leave the region, the mass of the fluid region remains constant.

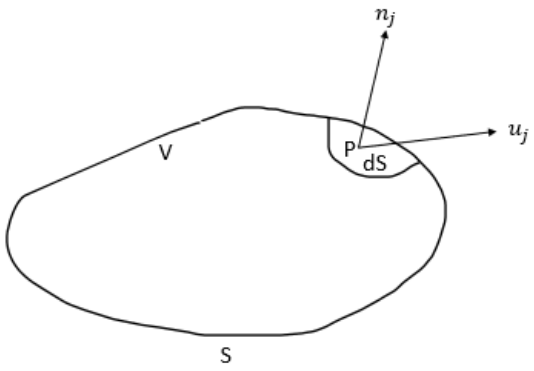


Figure 2.1: Fluid element bounding volume V.

2.1.1 Mathematical form of law of conservation of mass – Continuity equation

Let us consider a closed surface S enclosing a fixed volume V in a region occupied by moving fluid at time t. Let P be the position of any fluid particle in S. If n_j is the components of unit vector in the direction of the outward drawn normal to the surface S and u_j is the components of velocity of the fluid at that point, then the inward normal velocity is $-n_j u_j$.
The mass of the fluid entering per unit time through the element dS is given by

$$\rho(-n_j u_j)dS, \tag{2.1}$$

where ρ is the density of the fluid.
Thus, the total mass of the fluid entering the surface S per unit time is

$$-\int_S \rho(n_j u_j)dS. \tag{2.2}$$

The mass of the fluid within the surface S is

$$\int_V \rho dV.$$

Therefore, the rate at which the mass increases is

$$\frac{d}{dt}\int_V \rho dV = \int_V \frac{\partial \rho}{\partial t}dV. \tag{2.3}$$

By using the law of conservation of mass, from (2.2) and (2.3), we get

$$\int_V \frac{\partial \rho}{\partial t}dV = -\int_S \rho u_j n_j dS. \tag{2.4}$$

Due to Gauss divergence theorem, (2.4) becomes

$$\int_V \frac{\partial \rho}{\partial t}dV = -\int_V \frac{\partial}{\partial x_j}(\rho u_j)dV,$$

which implies

$$\int_V \left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}\right)(\rho u_j)dV = 0. \tag{2.5}$$

The equality (2.5) holds for arbitrary volume V; and hence we have

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0. \tag{2.6}$$

The equation (2.6) is known as continuity equation.

The vector form of (2.6) can be expressed as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0. \tag{2.7}$$

Another equivalent form of (2.7) is given by

$$\frac{D\rho}{Dt} + \rho \ \mathrm{div}(\vec{u}) = 0, \tag{2.8}$$

where

$$\frac{D(.)}{Dt} = \frac{\partial(.)}{\partial t} + (\vec{u} \cdot \nabla)(.).$$

For steady flow the continuity equation becomes

$$\mathrm{div}(\rho \vec{u}) = 0, \tag{2.9}$$

and for an incompressible fluid (density ρ is constant), it becomes

$$\mathrm{div}(\vec{u}) = 0. \tag{2.10}$$

2.2 Law of conservation of linear momentum

The principle of conservation of linear momentum may be regarded as Newton's second law of motion applied to a fluid. It states that the time rate of change of momentum of a body is equal to the sum of forces acting on the body.

2.2.1 Mathematical form of law of conservation of linear momentum – Navier-Stokes equations

Let us consider a closed surface S enclosing volume V in the region occupied by the moving fluid. Let u_j be the velocity components of the fluid and n_j is the components of outward-drawn normal

vector. The total linear momentum of the fluid entering through the surface S is

$$-\int_S v_i(\rho n_j u_j)dS. \tag{2.11}$$

The local rate of change of linear momentum in the closed volume V is

$$\frac{\partial}{\partial t}\int_V (\rho u_i)dV. \tag{2.12}$$

Therefore, the total rate of change of linear momentum of the fluid contained in V is given by

$$\frac{\partial}{\partial t}\int_V (\rho u_i)dV + \int_S u_i(\rho n_j u_j)dS. \tag{2.13}$$

In fluid motion, there are two types of force:

- **Body force**: Acts throughout the volume of the fluid, i.e., gravity, electric, and magnetic forces.
- **Surface force**: Acts on the boundary of the fluid, i.e., pressure and viscous forces.

Let f_i denote the components of body force per unit mass and P_i denote the components of surface force on the boundary per unit area. Then, the total body force acting through the volume V is given by

$$\int_V (\rho f_i)dV. \tag{2.14}$$

The total surface force acting on the surface S is

$$\int_S P_i\, dS. \tag{2.15}$$

Therefore, the mathematical expression corresponding to the law of conservation of linear momentum, taking into account (2.13)-(2.15), is given by

$$\frac{\partial}{\partial t}\int_V (\rho u_i)dV + \int_S u_i(\rho u_j n_j)dV = \int_V (\rho f_i)dV + \int_S P_i\, dS. \tag{2.16}$$

By Cauchy's law [11],

$$P_i = \sigma_{ij} n_j, \tag{2.17}$$

where σ_{ij} is the component of stress tensor. The first subscript in σ_{ij} indicates that this stress component acts on the plane $x_i =$ constant and the second subscript indicates that it acts in the x_j direction. Using the equation (2.17) and Gauss divergence theorem in equation (2.16), we obtain

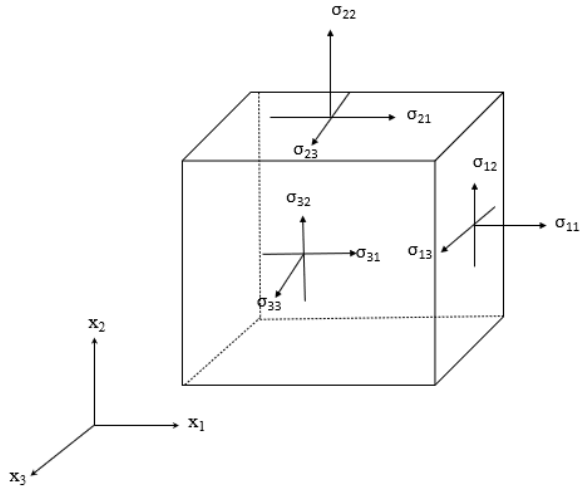


Figure 2.2: Representation of the nine components of stress tensor at a point in a fluid.

$$\frac{\partial}{\partial t}\int_V (\rho u_i) dV + \int_V \frac{\partial}{\partial x_j}(u_i \rho u_j) dV = \int_V (\rho f_i) dV + \int_V \frac{\partial \sigma_{ij}}{\partial x_j} dV,$$

$$\implies \int_V \left[\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(u_i \rho u_j) - (\rho f_i) - \frac{\partial \sigma_{ij}}{\partial x_j} \right] dV = 0.$$

Since V is arbitrary, we get

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(u_i \rho u_j) - (\rho f_i) - \frac{\partial \sigma_{ij}}{\partial x_j} = 0.$$

Using the equation of continuity (2.6), the above equation becomes

$$\rho \frac{\partial u_i}{\partial t} + (\rho u_j) \frac{\partial u_i}{\partial x_j} = \rho f_i + \frac{\partial \sigma_{ij}}{\partial x_j}. \tag{2.18}$$

Equation (2.18) is called Cauchy's equation of motion. From equation (2.18), we derive the Navier-Stokes equations, which are momentum equations for Newtonian fluid.

Newton's law of viscosity: It states that the shear stress between adjacent fluid layers is linearly proportional to the velocity gradient between the layers.

A fluid that follows Newton's law of viscosity is called Newtonian fluid. The equation of motion for the Newtonian fluid yields the celebrated Navier-Stokes equations, which are the fundamental equations governing fluid flow. For the Newtonian fluid, the constitutive relations are given by

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij} - \frac{2}{3}\mu e_{kk}\delta_{ij}, \tag{2.19}$$

where δ_{ij} is Kronecker delta and

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Using equation (2.19) in equation (2.18), we get

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\mu e_{ij} - \frac{2}{3}\mu e_{kk}\delta_{ij}\right),$$

which implies

$$\rho\frac{\partial u_i}{\partial t} + (\rho u_j)\frac{\partial u_i}{\partial x_j} = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) \tag{2.20}$$

The equation (2.20) is known as the Navier-Stokes equations.

When the fluid is incompressible and the viscosity is constant, the above equation (2.20) turns out to be

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = \rho f_i - \frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j^2}. \tag{2.21}$$

In vector form, the above equation becomes

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right) = \rho\vec{f} - \nabla p + \mu\nabla^2\vec{u}. \tag{2.22}$$

Chapter 3

Squeeze film lubrication between parallel plates

In this chapter, we present a study of the squeeze film flow of a Newtonian fluid between parallel plates under lubrication approximations.

Consider an incompressible Newtonian fluid between two infinitely long parallel plates separated by distance $h(t)$, where the lower plate is fixed and the upper plate moves downward with velocity $\frac{dh}{dt}$. The equations governing the flow, in the absence of body forces, are given by

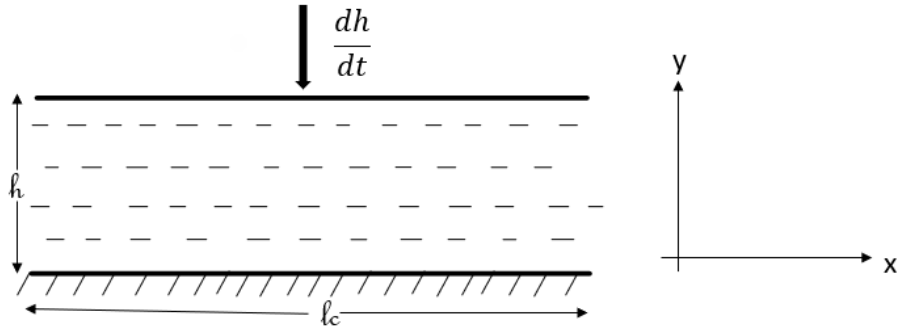


Figure 3.1: Geometry of the problem

$$\nabla \cdot \vec{u} = 0. \tag{3.1}$$

and

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = -\nabla \tilde{p} + \mu \nabla^2 \vec{u}, \tag{3.2}$$

where $\vec{u}$ is the fluid velocity, ρ is the fluid density, μ is viscosity coefficient, and $\tilde{p}$ is the fluid pressure.

3.1 Mathematical formulation

Since the flow is between infinitely long parallel plates, it is two dimensional; therefore, we take $\vec{u} = (\tilde{u}, \tilde{v})$. Now, the equations (3.1) and (3.2) turn out be

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0, \tag{3.3}$$

***x*-momentum equation**:

$$\rho\left(\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}}\right) = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \mu\left(\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right), \tag{3.4}$$

***y*-momentum equation**:

$$\rho\left(\frac{\partial \tilde{v}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}}\right) = -\frac{\partial \tilde{p}}{\partial \tilde{y}} + \mu\left(\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}\right). \tag{3.5}$$

The boundary conditions to be satisfied are

$$\left.\begin{aligned} \tilde{u} = 0 \text{ and } \tilde{v} = 0 \quad &\text{at } \tilde{y} = 0, \\ \tilde{u} = 0 \text{ and } \tilde{v} = \frac{dh}{d\tilde{t}} \quad &\text{at } \tilde{y} = h, \\ \frac{\partial \tilde{p}}{\partial \tilde{x}} = 0 \quad \text{at } \tilde{x} = 0, \text{ and } \tilde{p} = p_e \quad &\text{at } \tilde{x} = l_c. \end{aligned}\right\} \tag{3.6}$$

Using the non-dimensional variables

$$x = \frac{\tilde{x}}{l_c}, \quad y = \frac{\tilde{y}}{h_0}, \quad t = \frac{v_c \tilde{t}}{h_0}, \quad u = \frac{h_0 \tilde{u}}{l_c v_c}, \quad v = \frac{\tilde{v}}{v_c}, \quad H = \frac{h}{h_0}, \quad p = \frac{h_0^3(\tilde{p} - p_e)}{\mu v_c l_c^2},$$

equations (3.3), (3.4) and (3.5) become, respectively,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.7}$$

$$Re\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \varepsilon^2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \tag{3.8}$$

and

$$Re\,\varepsilon^2\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial P}{\partial y} + \varepsilon^4\frac{\partial^2 v}{\partial x^2} + \varepsilon^2\frac{\partial^2 v}{\partial y^2}, \tag{3.9}$$

where $Re = \frac{\rho v_c h_0}{\mu}$ is squeeze Reynolds number, $\varepsilon = \frac{h_0}{l_c}$, h_0 is the initial position of the upper plate and v_c is characteristic velocity.

In non-dimensional form, boundary conditions (3.6) become

$$\left.\begin{aligned} u = 0 \text{ and } v = 0 \quad &\text{at } y = 0, \\ u = 0 \text{ and } v = \frac{dH}{dt} \quad &\text{at } y = H, \\ \frac{\partial p}{\partial x} = 0 \quad \text{at } x = 0, \text{ and } p = 1 \quad &\text{at } x = 1. \end{aligned}\right\} \tag{3.10}$$

In view of lubrication approximation (i.e., $h_0 << l_c$), we have $\varepsilon << 1$; therefore, neglecting terms involving ε, the equations (3.7)-(3.9) become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.11}$$

$$Re\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \tag{3.12}$$

$$0 = -\frac{\partial p}{\partial y}. \tag{3.13}$$

In view of (3.13), p is independent of y.

3.2 Solution

It can be observed that the system of partial differential equations (PDEs) (3.11) and (3.12) is coupled in u, v, p and non-linear; however, (3.12) is decoupled from v. We seek the solution of the system of PDEs (3.11) and (3.12) in the form

$$u = u_0 + Reu_1, \tag{3.14}$$

$$v = v_0 + Rev_1, \tag{3.15}$$

$$p = p_0 + Rep_1. \tag{3.16}$$

Zeroth order system and its solution:

Using (3.14)-(3.16) in (3.11) and (3.12), and comparing the coefficients of Re^0, we get the zeroth order system as

$$\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}=0, \tag{3.17}$$

$$-\frac{\partial p_0}{\partial x}+\frac{\partial^2 u_0}{\partial y^2}=0. \tag{3.18}$$

The corresponding boundary conditions are

$$\left.\begin{aligned} u_0=0 \text{ and } v_0=0 \quad \text{at } y=0,\\ u_0=0 \text{ and } v_0=\frac{dH}{dt} \quad \text{at } y=H,\\ \frac{\partial p_0}{\partial x}=0 \quad \text{at } x=0, \text{ and } p_0=1 \quad \text{at } x=1. \end{aligned}\right\} \tag{3.19}$$

Integrating (3.18) with respect to y, we get

$$u_0=\frac{\partial p_0}{\partial x}\frac{y^2}{2}+C_1 y+C_2. \tag{3.20}$$

Using the boundary condition $u_0=0$ at $y=0$ in (3.20), we get

$$C_2=0,$$

and another boundary condition $u_0=0$ at $y=H$ gives

$$C_1=-\frac{H}{2}\frac{\partial p_0}{\partial x}.$$

Therefore, we get

$$u_0=\frac{1}{2}\frac{\partial p_0}{\partial x}\left(y^2-Hy\right). \tag{3.21}$$

From equation (3.17), in view of (3.21), we get

$$\frac{\partial v_0}{\partial y}=-\frac{\partial u_0}{\partial x}=-\frac{\partial^2 p_0}{\partial x^2}\left(\frac{y^2}{2}-\frac{Hy}{2}\right).$$

Integrating with respect to y, the expression for v_0 is obtained as

$$v_0=-\frac{\partial^2 p_0}{\partial x^2}\left(\frac{y^3}{6}-\frac{Hy^2}{4}\right)+C_3. \tag{3.22}$$

Using the boundary condition $v_0 = 0$ at $y = 0$, we get

$$C_3 = 0.$$

Putting $C_3 = 0$ in (3.22) and using the boundary condition $v_0 = \frac{dH}{dt}$ at $y = H$, we get

$$\frac{\partial^2 p_0}{\partial x^2} = \frac{12}{H^3}\frac{dH}{dt}. \tag{3.23}$$

The equation (3.23) is known as classical Reynolds equation.

Now using (3.23) in (3.22), the solution for v_0 turn out to be

$$v_0 = -\frac{6}{H^3}\frac{dH}{dt}\left(\frac{y^3}{3} - \frac{Hy^2}{2}\right). \tag{3.24}$$

Integrating the equation (3.23) with respect to x, we get

$$\frac{\partial p_0}{\partial x} = \frac{12}{H^3}\frac{dH}{dt}x + C_4.$$

The boundary conditions $\frac{\partial p_0}{\partial x} = 0$ at $x = 0$, gives

$$C_4 = 0,$$

and hence

$$\frac{\partial p_0}{\partial x} = \frac{12}{H^3}\frac{dH}{dt}x. \tag{3.25}$$

Integrating (3.25) with respect to x, we get

$$p_0 = \frac{12}{H^3}\frac{dH}{dt}\frac{x^2}{2} + C_5. \tag{3.26}$$

Using boundary conditions $p_0 = 0$ at $x = 0$ in (3.26), we get

$$p_0 = -\frac{6}{H^3}\frac{dH}{dt}x^2 + \frac{6}{H^3}\frac{dH}{dt}. \tag{3.27}$$

Putting (3.25) in (3.20), we get

$$u_0 = \frac{6x}{H^3}\frac{dH}{dt}\left(y^2 - Hy\right). \tag{3.28}$$

Therefore, the zeroth order solutions are

$$u_0 = \frac{6x}{H^3}\frac{dH}{dt}\left(y^2 - Hy\right), \tag{3.29}$$

$$v_0 = -\frac{6}{H^3}\frac{dH}{dt}\left(\frac{y^3}{3} - \frac{Hy^2}{2}\right), \tag{3.30}$$

and

$$p_0 = -\frac{6}{H^3}\frac{dH}{dt}\left(x^2 - 1\right). \tag{3.31}$$

First order system and its solution:

Using (3.14)-(3.16) in (3.11) and (3.12), and comparing the coefficients of Re^1, we get the first order system as

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \tag{3.32}$$

$$-\frac{\partial p_1}{\partial x} + \frac{\partial^2 u_1}{\partial y^2} = \frac{\partial u_0}{\partial t} + u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y}. \tag{3.33}$$

The corresponding boundary conditions are given by

$$\left.\begin{aligned} u_1 = 0 \text{ and } v_1 = 0 \quad \text{at } y = 0, \\ u_1 = 0 \text{ and } v_1 = 0 \quad \text{at } y = H, \\ \frac{\partial p_1}{\partial x} = 0 \quad \text{at } x = 0, \text{ and } p_1 = 0 \quad \text{at } x = 1. \end{aligned}\right\} \tag{3.34}$$

Substituting the expression of u_0 and v_0, obtained in the above subsection, in equation (3.33), we get

$$\begin{aligned} -\frac{\partial p_1}{\partial x} + \frac{\partial^2 u_1}{\partial y^2} = {} & \frac{\partial}{\partial t}\left(\frac{6x}{H^3}\frac{dH}{dt}\right)(y^2 - Hy) + \left(\frac{6x}{H^3}\frac{dH}{dt}\right)(y^2 - Hy)\frac{\partial}{\partial x}\left(\frac{6x}{H^3}\frac{dH}{dt}(y^2 - Hy)\right) \\ & + \left(-\frac{6}{H^3}\frac{dH}{dt}\left(\frac{y^3}{3} - \frac{Hy^2}{2}\right)\right)\frac{\partial}{\partial y}\left(\frac{6x}{H^3}\frac{dH}{dt}(y^2 - Hy)\right). \end{aligned}$$

$$\begin{aligned} \implies \frac{\partial^2 u_1}{\partial y^2} = {} & -\frac{18x}{H^4}\left(\frac{dH}{dt}\right)^2(y^2 - Hy) + \frac{6x}{H^3}\frac{d^2H}{dt^2}(y^2 - Hy) - \frac{6xy}{H^3}\left(\frac{dH}{dt}\right)^2 \\ & + \frac{36x}{H^6}\left(\frac{dH}{dt}\right)^2\left(y^4 + H^2y^2 - 2y^3H\right) - \frac{36x}{H^6}\left(\frac{dH}{dt}\right)^2\left(\frac{2y^4}{3} - \frac{4Hy^3}{3} + \frac{y^2H^2}{2}\right) + \frac{\partial p_1}{\partial x}. \end{aligned} \tag{3.35}$$

Integrating the above equation with respect to y, we get

$$\frac{\partial u_1}{\partial y} = \frac{\partial p_1}{\partial x} y - \frac{18x}{H^4}\left(\frac{\partial H}{\partial t}\right)^2\left(\frac{y^3}{3} - \frac{Hy^2}{2}\right) + \frac{6x}{H^3}\frac{\partial^2 H}{\partial t^2}\left(\frac{y^3}{3} - \frac{Hy^2}{2}\right) - \frac{3xy^2}{H^3}\left(\frac{\partial H}{\partial t}\right)^2 + \frac{36x}{H^6}\left(\frac{\partial H}{\partial t}\right)^2\left(\frac{y^5}{15} + \frac{H^2y^3}{6} - \frac{Hy^4}{6}\right) + C_6. \quad (3.36)$$

Again integrating (3.36) with respect to y, we get

$$u_1 = \frac{\partial p_1}{\partial x}\frac{y^2}{2} - \frac{18x}{H^4}\left(\frac{dH}{dt}\right)^2\left(\frac{y^4}{12} - \frac{Hy^3}{6}\right) + \frac{6x}{H^3}\frac{d^2H}{dt^2}\left(\frac{y^4}{12} - \frac{Hy^3}{6}\right) - \frac{xy^3}{H^3}\left(\frac{\partial H}{\partial t}\right)^2 + \frac{36x}{H^6}\left(\frac{\partial H}{\partial t}\right)^2\left(\frac{y^6}{90} - \frac{Hy^5}{30} + \frac{H^2y^4}{24}\right) + C_6 y + C_7 \quad (3.37)$$

Using boundary condition $u_1 = 0$ at $y = 0$ in equation (3.37), we obtain

$$C_7 = 0,$$

and the boundary condition $u_1 = 0$ at $y = H$, gives

$$C_6 = -\frac{\partial p_1}{\partial x}\frac{H}{2} + \frac{18x}{H^4}\left(\frac{dH}{dt}\right)^2\left(-\frac{H^3}{12}\right) + \frac{6x}{H^3}\frac{d^2H}{dt^2}\left(\frac{H^3}{12}\right) + \frac{x}{H}\left(\frac{\partial H}{\partial t}\right)^2 - \frac{36x}{H^6}\left(\frac{\partial H}{\partial t}\right)^2\left(\frac{7H^5}{360}\right).$$

Therefore, the solution for u_1 is simplified as

$$u_1 = \frac{\partial p_1}{\partial x}\frac{y^2}{2} - \frac{18x}{H^4}\left(\frac{dH}{dt}\right)^2\left[\frac{y^4}{12} - \frac{Hy^3}{6}\right] + \frac{6x}{H^3}\frac{d^2H}{dt^2}\left[\frac{y^4}{12} - \frac{Hy^3}{6}\right] - \frac{xy^3}{H^3}\left(\frac{\partial H}{\partial t}\right)^2 + \frac{36x}{H^6}\left(\frac{\partial H}{\partial t}\right)^2\left[\frac{y^6}{90} - \frac{Hy^5}{30} + \frac{H^2y^4}{24}\right] - \frac{\partial p_1}{\partial x}\frac{Hy}{2} - \frac{6xy}{5H}\left(\frac{dH}{dt}\right)^2 + \frac{xy}{2}\frac{d^2H}{dt^2}. \quad (3.38)$$

From (3.32), we have

$$\frac{\partial v_1}{\partial y} = -\frac{\partial u_1}{\partial x};$$

consequently, we get

$$v_1 = -\frac{\partial^2 p_1}{\partial x^2}\frac{y^3}{6} + \frac{18}{H^4}\left(\frac{dH}{dt}\right)^2\left[\frac{y^5}{60} - \frac{Hy^4}{24}\right] - \frac{6}{H^3}\frac{d^2H}{dt^2}\left[\frac{y^5}{60}\frac{Hy^4}{24}\right] + \frac{y^4}{4H^3}\left(\frac{\partial H}{\partial t}\right)^2 - \frac{36}{H^6}\left(\frac{\partial H}{\partial t}\right)^2\left[\frac{y^7}{630} - \frac{Hy^6}{180} + \frac{H^2y^5}{120}\right] + \frac{\partial^2 p_1}{\partial x^2}\frac{Hy^2}{4} + \frac{3y^2}{5H}\left(\frac{\partial H}{\partial t}\right)^2 - \frac{y^2}{4}\frac{\partial^2 H}{\partial t^2} + C_8. \quad (3.39)$$

Using boundary condition $v_1 = 0$ at $y = 0$ in equation (3.39), we get $C_8 = 0$, and hence the solution for v_1 becomes

$$v_1 = -\frac{\partial^2 p_1}{\partial x^2}\frac{y^3}{6} + \frac{18}{H^4}\left(\frac{dH}{dt}\right)^2\left[\frac{y^5}{60} - \frac{Hy^4}{24}\right] - \frac{6}{H^3}\frac{d^2H}{dt^2}\left[\frac{y^5}{60}\frac{Hy^4}{24}\right] + \frac{y^4}{4H^3}\left(\frac{\partial H}{\partial t}\right)^2 - \frac{36}{H^6}\left(\frac{\partial H}{\partial t}\right)^2\left[\frac{y^7}{630} - \frac{Hy^6}{180} + \frac{H^2y^5}{120}\right] + \frac{\partial^2 p_1}{\partial x^2}\frac{Hy^2}{4} + \frac{3y^2}{5H}\left(\frac{\partial H}{\partial t}\right)^2 - \frac{y^2}{4}\frac{\partial^2 H}{\partial t^2}. \quad (3.40)$$

Using the boundary condition $v_1 = 0$ at $y = H$ in (3.40), we get

$$\frac{\partial^2 p_1}{\partial x^2} = -\frac{102}{35H^2}\left(\frac{\partial H}{\partial t}\right)^2 + \frac{6}{5H}\frac{\partial^2 H}{\partial t^2}. \quad (3.41)$$

This equation is known as modified Reynolds equation.

Integrating (3.41), we get

$$p_1 = -\frac{102}{35H^2}\frac{x^2}{2}\left(\frac{\partial H}{\partial t}\right)^2 + \frac{6x^2}{10H}\frac{\partial^2 H}{\partial t^2} + C_9 x + C_{10}. \quad (3.42)$$

After using the appropriate boundary conditions for p_1, (3.42) becomes

$$p_1 = \frac{51}{35H^2}\left(\frac{dH}{dt}\right)^2(1-x^2) - \frac{3}{5H}\frac{d^2H}{dt^2}(1-x^2). \quad (3.43)$$

In view of (3.43), the expression for u_1 and v_1 can be written as, respectively,

$$u_1 = \left(\frac{-102x}{35H^2}\left(\frac{dH}{dt}\right)^2 + \frac{6x}{5H}\frac{d^2H}{dt^2}\right)\left(\frac{y^2}{2} - \frac{Hy}{2}\right) - \frac{18x}{H^4}\left(\frac{dH}{dt}\right)^2\left(\frac{y^4}{12} - \frac{Hy^3}{6}\right) + \frac{6x}{H^3}\frac{d^2H}{dt^2}\left(\frac{y^4}{12} - \frac{Hy^3}{6}\right) - \frac{xy^3}{H^3}\left(\frac{dH}{dt}\right)^2 + \frac{36x}{H^6}\left(\frac{dH}{dt}\right)^2\left(\frac{y^6}{90} + \frac{H^2y^4}{24} - \frac{Hy^5}{30}\right) - \frac{6}{5H}xy\left(\frac{dH}{dt}\right)^2 + \frac{xy}{2}\frac{d^2H}{dt^2}. \quad (3.44)$$

and

$$v_1 = \left(\frac{-102}{35H^2}\left(\frac{dH}{dt}\right)^2 + \frac{6}{5H}\frac{d^2H}{dt^2}\right)\left(\frac{Hy^2}{4} - \frac{y^3}{6}\right) + \frac{18}{H^4}\left(\frac{dH}{dt}\right)^2\left(\frac{y^5}{60} - \frac{Hy^4}{24}\right) - \frac{6}{H^3}\frac{d^2H}{dt^2}\left(\frac{y^5}{60} - \frac{Hy^4}{24}\right) + \frac{y^4}{4H^3}\left(\frac{dH}{dt}\right)^2 - \frac{36}{H^6}\left(\frac{dH}{dt}\right)^2\left(\frac{y^7}{630} + \frac{H^2y^5}{630} - \frac{Hy^6}{180}\right) + \frac{3y^2}{5H}\left(\frac{dH}{dt}\right)^2 - \frac{y^2}{4}\frac{d^2H}{dt^2}. \quad (3.45)$$

Now, the analytical solutions, taking into account the zeroth and first order solutions, is given by

$$u = u_0 + Re\ u_1, \tag{3.46}$$

$$v = v_0 + Re\ v_1, \tag{3.47}$$

and

$$p = p_0 + Re\ p_1, \tag{3.48}$$

where the expression of u_0, v_0, p_0 and u_1, v_1, p_1 are given by equations (3.29), (3.30), (3.31), (3.44), (3.45) and (3.43), respectively.

Load carrying capacity:

In lubrication theory, the load carrying capacity refers to the maximum amount of load (force) a lubricating film can withstand and support between two moving surfaces without causing direct contact and wear. Essentially, it represents the ability of the lubricant to separate the surfaces and prevent friction by generating pressure within the fluid film.

The load carrying capacity on the upper plate (f_L) per unit width can be calculated by integrating the fluid pressure over the unit film length, that is,

$$\begin{aligned} f_L &= \int_0^1 p\ dx \\ &= -\frac{4}{H^3}\frac{dH}{dt} + Re\left[\frac{34}{35H^2}\left(\frac{dH}{dt}\right)^2 - \frac{2}{5H}\frac{d^2H}{dt^2}\right]. \end{aligned} \tag{3.49}$$

3.3 Discussion of results

The squeeze film motion of a Newtonian fluid between two parallel plates is analyzed. Analytical expressions for the pressure distribution and the load-carrying capacity are derived. From the governing equation (3.12), it is observed that as the squeezing Reynolds number $\left(Re = \frac{\rho v_c h_0}{\mu}\right)$ approaches zero, the two-dimensional Navier–Stokes equations reduce to an inertia-free model. The solution of governing equations is sought in terms of asymptotic expansion, up to the first order, in terms of squeeze film Reynolds number. Initially, the zeroth-order problem is solved, which represents the classical Reynolds equation(s) that can be also derived under the assumption of negligible inertia. Subsequently, the first-order inertial correction of the pressure is obtained from the modified Reynolds equation.

In the present analysis, the results for the pressure and the load carrying capacity are presented for different values of Re and t, considering two cases of the time-height relation [12, 13].

Case 1: $H(t) = (1-t)$.

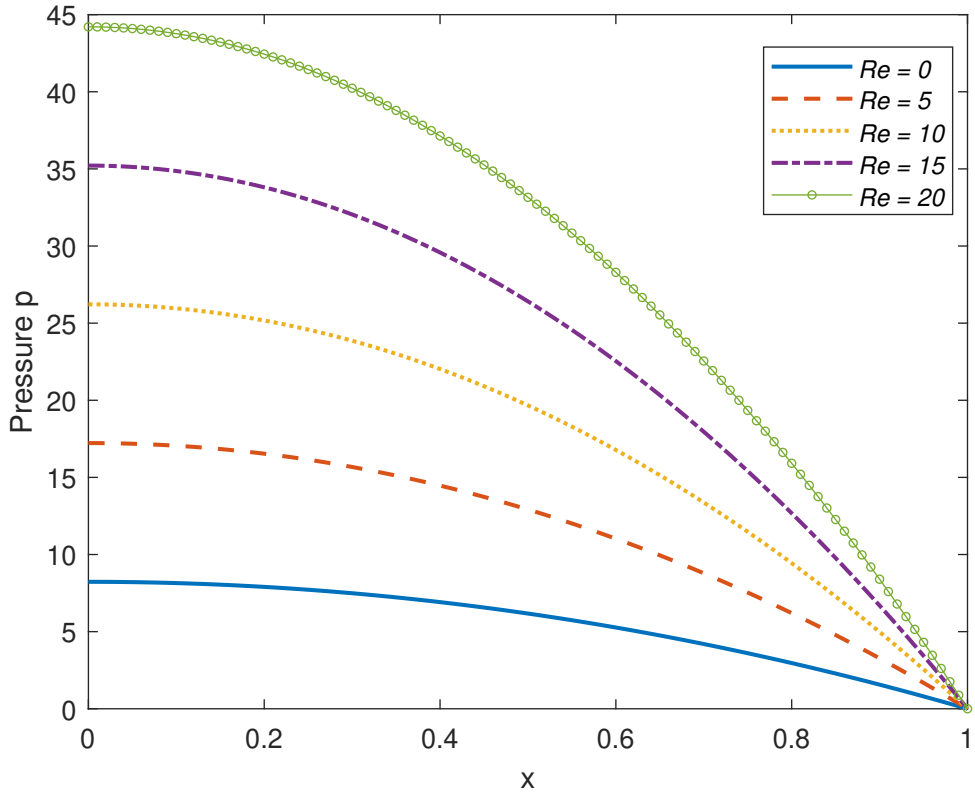


Figure 3.2: Pressure distribution $p(x)$ vs x at $t = 0.1$ for different values of Re.

Figure 3.2 describes the variation of film pressure p with coordinate x for different values of Reynolds number Re at $t = 0.1$. An increment in p is noticed for a higher Reynolds number, which indicates that the convective inertia has an increasing effect on the film pressure.

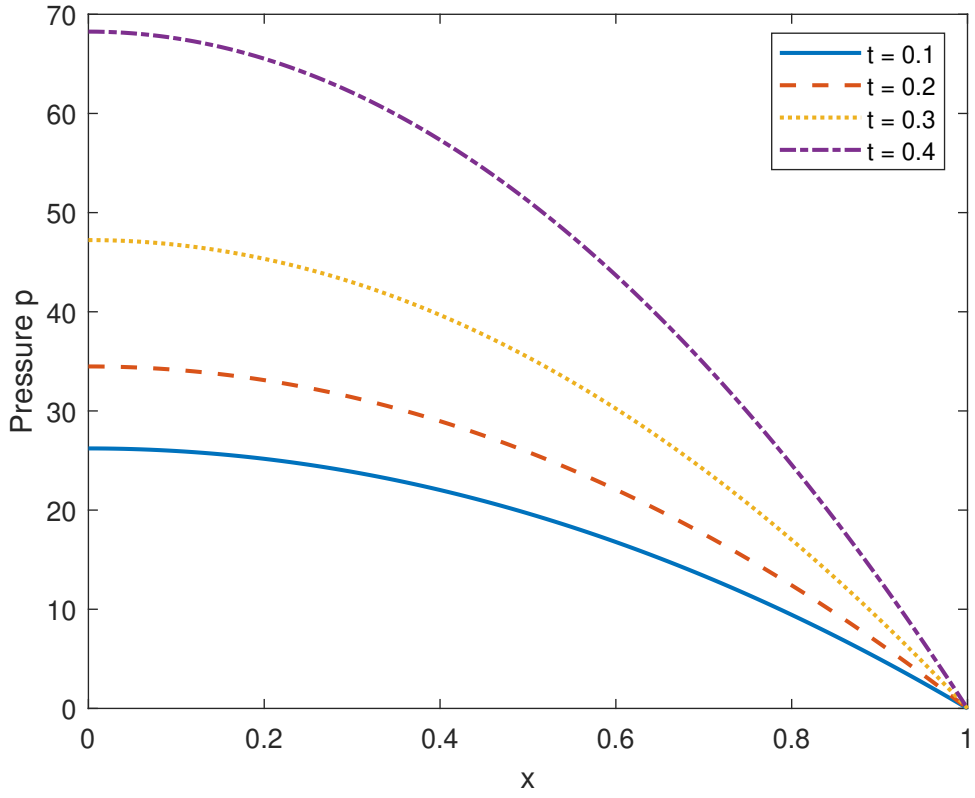


Figure 3.3: Pressure distribution $p(x)$ vs x for different values of t when $Re = 10$.

Figure 3.3 describes the variation of film pressure p with coordinate x for different values of t when the Reynolds number $Re = 10$. There is an increase in film pressure with increasing values of t; thus, larger increments of the pressure are obtained for a smaller film thickness H.

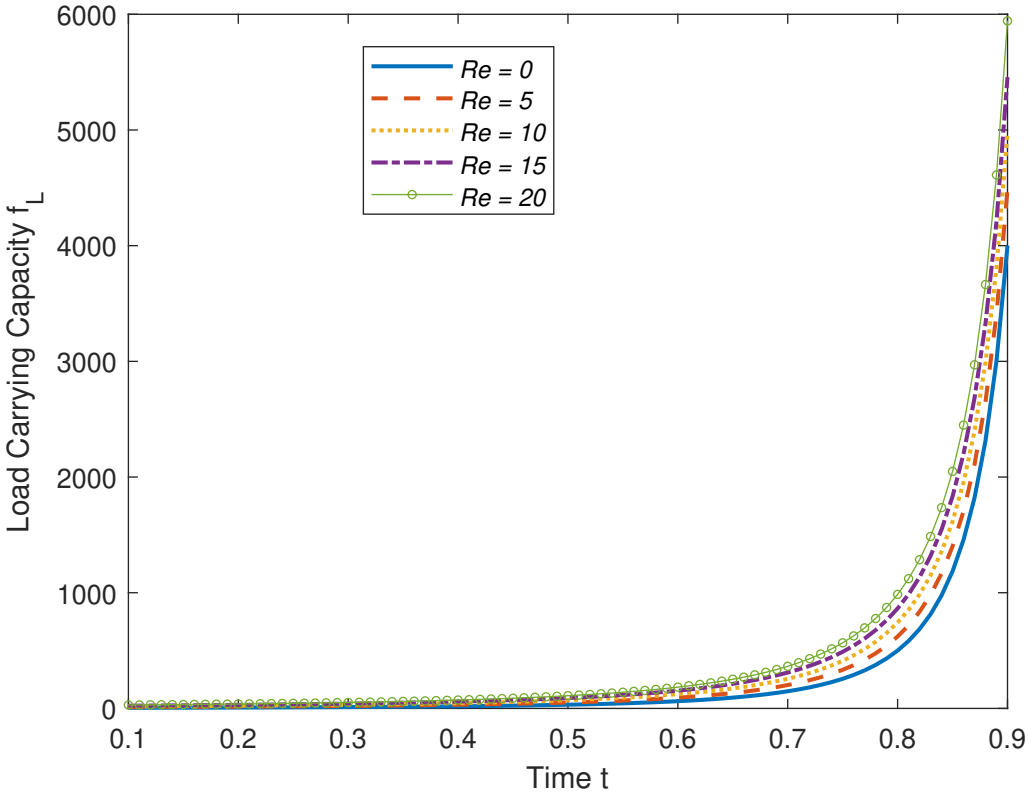


Figure 3.4: Load carrying capacity f_L vs t for different values of Re.

Figure 3.4 displays the variation of the load carrying capacity (f_L) per unit width with time t, for different values Re. Since inertial effect results in a higher squeeze film pressure, the load carrying capacity is expected to be similarly affected. As observed in the figure, higher values of the load carrying capacity are obtained for larger values of squeezing Reynolds number (Re).

Case 2: $H(t) = \sqrt{1-2t}, \quad t < 0.5.$

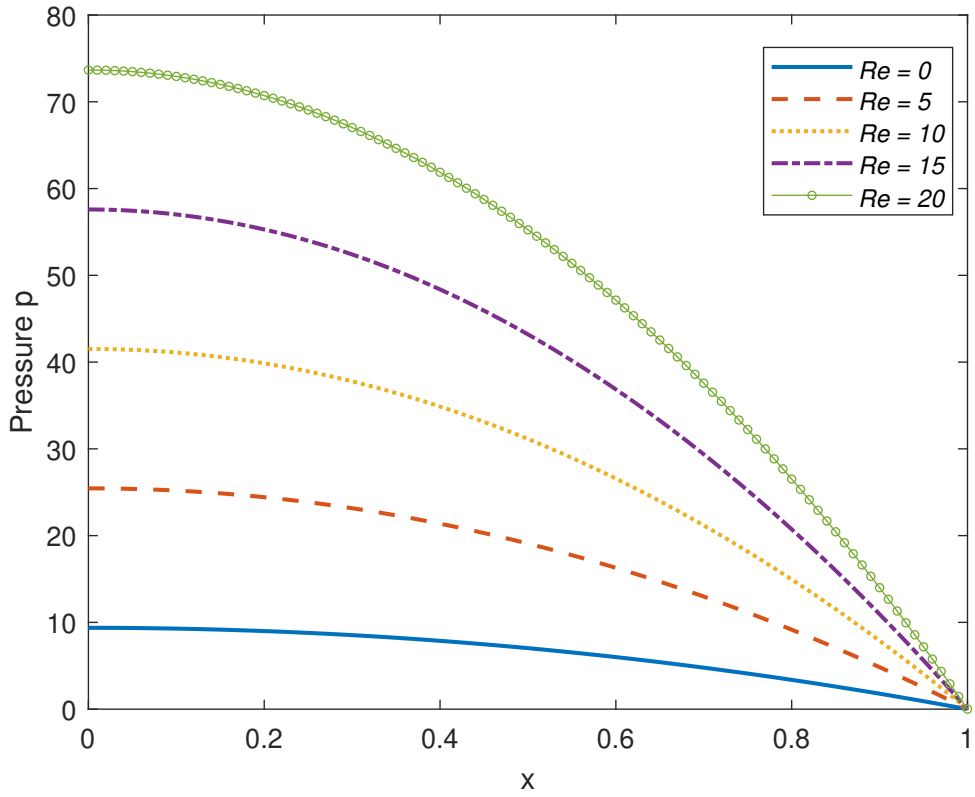


Figure 3.5: Pressure distribution $p(x)$ vs x at $t = 0.1$ for different values of Re.

Figure 3.5 depicts the variation of pressure with coordinate x for different values of squeezing Reynolds number Re at $t = 0.1$. Increment in pressure p is seen for increasing values of Re.

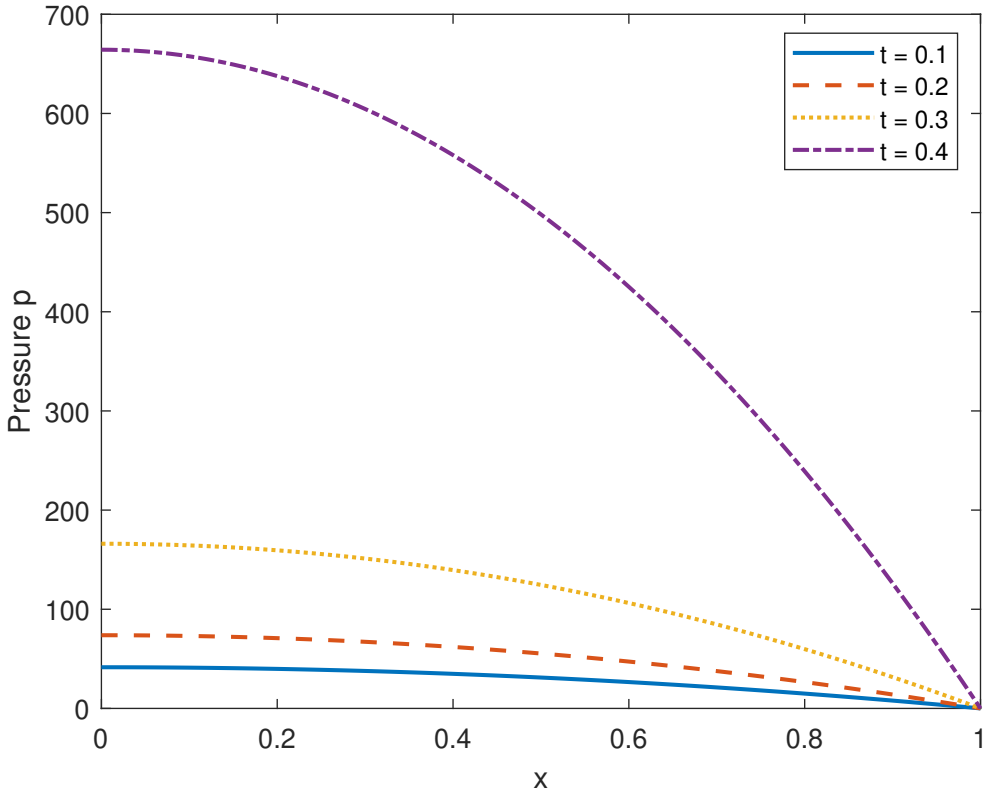


Figure 3.6: Pressure distribution $p(x)$ vs x for different values of t when $Re = 10$.

Figure 3.6 illustrates the variation of film pressure p along the coordinate x for different values of time t at a fixed Reynolds number $Re = 10$. It is observed that the film pressure increases with increasing t.

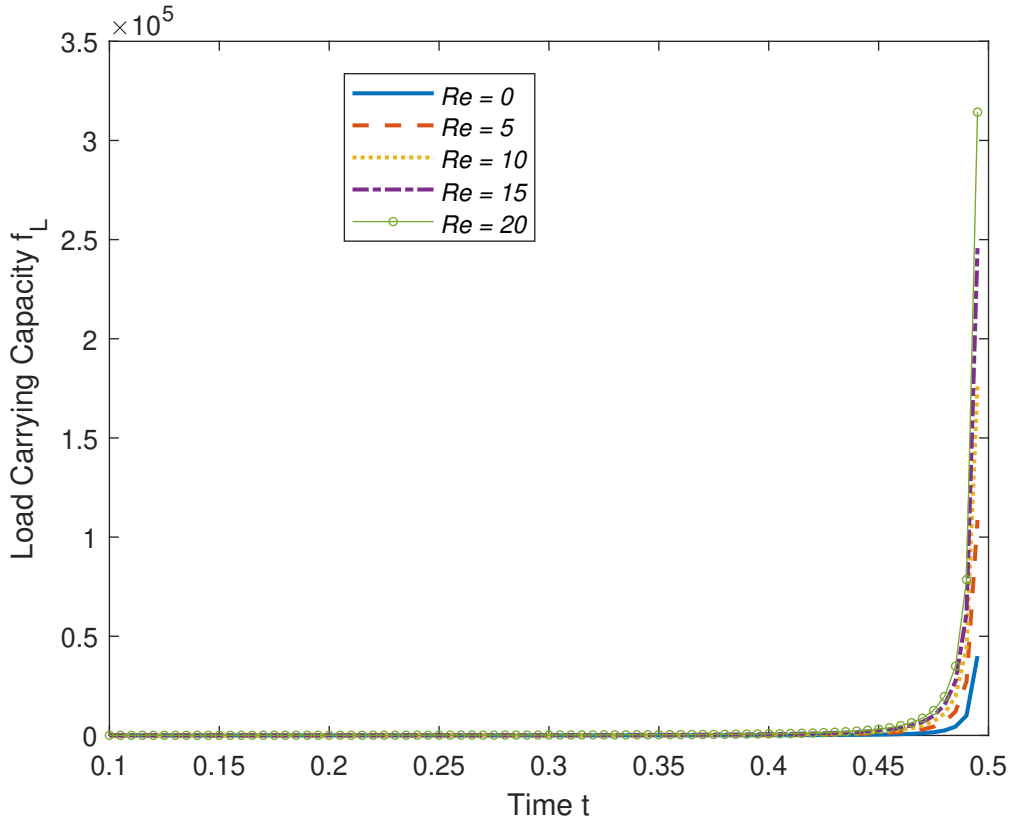


Figure 3.7: Load carrying capacity f_L vs t for different values of Re.

Figure 3.7 presents the variation of the load carrying capacity per unit width (f_L) with time t, for different values of Re. As observed in the figure, the load carrying capacity increases with increasing Re.

It is evident from the above discussion that the pressure distribution in Case 2 is higher than in Case 1. In Case 1, the film thickness is given by $H(t) = 1 - t$, which decreases linearly with time. In contrast, Case 2 involves a nonlinear film thickness profile, $H(t) = \sqrt{1-2t}$, which decreases more rapidly as $t \to 0.5$. From the expression (3.48), it is observed that pressure p is inversely proportional to the thickness of the film $H(t)$; consequently, a rapid decrease in the thickness of the film $H(t)$ in Case 2 leads to a significantly higher pressure distribution compared to that of Case 1.

3.4 Conclusions

The effect of fluid inertia forces on the squeeze film motion of a Newtonian fluid between two parallel plates has been analyzed. The expressions for the squeeze film pressure and the load-carrying capacity have been found analytically. The following conclusions are drawn:

- The squeezing Reynolds number (Re) significantly affects the squeeze film motion. Compared to the inertia-free case, an increase in Re leads to a higher film pressure and an enhanced load carrying capacity.
- As the distance between the two parallel plates decreases, the pressure distribution becomes more pronounced, leading to an increase in the load carrying capacity.

Chapter 4

Conclusions

In this dissertation, mathematical equations concerning the fundamental laws governing fluid motion were systematically derived, and were then applied to analyze squeeze film lubrication between parallel plates retaining the inertia terms. The inclusion of inertia extends the classical Reynolds theory and leads to a richer understanding of pressure development and load capacity under varying flow conditions.

The analytical solution revealed that the fluid inertia significantly affects the pressure distribution, especially at higher Reynolds numbers. As the Reynolds number increases, the peak pressure shifts and the load-carrying capacity exhibits notable changes. These findings emphasize the importance of considering inertial effects in the analysis and design of lubricated systems operating at high speeds or with rapidly varying film thickness.

This work contributes to the theoretical understanding of thin film flows and serves as a stepping stone for more advanced studies involving non-parallel geometries, time-dependent boundary motion, non-Newtonian effects in squeeze film lubrication etc.

Bibliography

[1] Hirani, H., *Fundamentals of engineering tribology with applications*, Cambridge University Press, 2016.

[2] Wu, H., *An analysis of the squeeze film between porous rectangular plates*, Journal of Lubrication Technology 94 (1972): 64-68.

[3] Ramanaiah, G., *Squeeze films between finite plates lubricated by fluids with couple stress*, Wear 54 (1979): 315-320.

[4] Naduvinamani, N. B., Fathima, S. T., and Hiremath, P. S., *Couple-stress squeeze films between porous rectangular plates*, Proceedings of the Institution of Mechanical Engineers, Part J: Journal of Engineering Tribology 217 (2003): 229-234.

[5] Lin, J. R., Liao, W. H., and Hung, C. R., *The effects of couple stresses in the squeeze film characteristics between a cylinder and a plane surface*, Journal of Marine science and technology 12 (2004): 7.

[6] Lin, J. R., *Magneto-hydrodynamic squeeze film characteristics for finite rectangular plates*, Industrial Lubrication and Tribology 55 (2003): 84-89.

[7] Bujurke, N. M., and Kudenatti, R. B., *MHD lubrication flow between rough rectangular plates*, Fluid dynamics research 39 (2007): 334.

[8] Kudenatti, R. B., Basti, D. P. and Bujurke, N. M., *Numerical solution of the MHD Reynolds equation for squeeze film lubrication between two parallel surfaces*, Applied Mathematics and computation 218 (2012): 9372-9382.

[9] Hanumagowda, B. N., Swapna, S. N., and Asha, B. S., *Analysis of effect of magneto-hydrodynamic, couple-stress and roughness on conical bearing*, Journal of Physics: Conference Series. Vol. 1000. No. 1. IOP Publishing, 2018.

[10] Patil, J., Naganagowda, H. B., Neela, D., Patil, V., Hiremath, A. G., Biradar, T. B., and Patil, V., *Analysis of MHD Effects on Porous Flat Plate and Curved Circular Plate with Couple Stress Fluid*, Journal of Advanced Research in fluid Mechanics and Thermal sciences 120 (2024): 82-98.

[11] Currie, I. G., *Fundamental mechanics of fluids*, CRC press, 2002.

[12] Phan-Thien, N., and Tanner, R. I., *Viscoelastic squeeze-film flows–Maxwell fluids*, Journal of Fluid Mechanics 129 (1983): 265-281.

[13] Khaled, A. R. A., and Vafai, K., *Analysis of flow and heat transfer inside nonisothermal squeezed thin films*, Numerical Heat Transfer, Part A: Applications 47 (2005): 981-996.